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Abstract

Scanning magnetic microscopes typically measure the vertical component B_3 of the magnetic field on a horizontal rectangular grid at close proximity to the sample. This feature makes them valuable instruments for analyzing magnetized materials at fine spatial scales, *e.g.*, in geosciences to access ancient magnetic records that might be preserved in rocks by mapping the external magnetic field generated by the magnetization within a rock sample. Recovering basic characteristics of the magnetization (such as its net moment, *i.e.*, the integral of the magnetization over the sample's volume) is an important problem, specifically when the field is too weak or the magnetization too complex to be reliably measured by standard bulk moment magnetometers.

In this paper, we establish formulas, asymptotically exact when R goes large, linking the integral of $x_1\,B_3$, $x_2\,B_3$, and B_3 over a square region of size R to the first, second, and third

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component of the net moment (and higher moments), respectively, of the magnetization generating B_3. The considered square regions are centered at the origin and have sides either parallel to the axes or making a 45-degree angle with them. Differences between the exact integrals and their approximations by these asymptotic formulas are explicitly estimated, allowing one to establish rigorous bounds on the errors.

We show how the formulas can be used for numerically estimating the net moment, so as to effectively use scanning magnetic microscopes as moment magnetometers. Illustrations of the method are provided using synthetic examples.

1 Introduction

Estimating the net moment of a magnetization distribution is a fundamental issue in studies of magnetized materials conducted in research areas ranging from geosciences and material sciences to medical imaging. Standard magnetometers infer the net moment from a set of measurements of the magnetic field taken at a fixed distance from the sample or integrated over a defined sensing area/volume. Typically, assumptions are made about the shape of the specimen under analysis and the uniformity of its magnetization so that simple and explicit formulas connect the magnetic dipole moment with the measured magnetic field or flux [4].

While such an approximation is valid under certain conditions (*e.g.*, the field is measured with sufficient accuracy far enough from the support of the magnetization; the magnetization is uniform and its support nearly spherical; the support of the magnetization is much smaller than the sensing region), it is not satisfactory when dealing with very weakly magnetized objects whose fields may get easily mixed with those of spurious magnetic sources away from the sample. Still, analyzing weak magnetization distributions is of considerable interest, *e.g.*, in paleomagnetism and biomagnetism. To this effect, scanning magnetic microscopes capable of measuring very weak fields at submillimetric distances from an object have been developed in recent years. These advances in instrumentation have driven the need for developing alternative techniques for estimating the net moment of a magnetization distribution from a set of magnetic field measurements.

From a mathematical viewpoint, the problem consists of recovering the mean of a compactly supported vector field from knowledge of the gradient of the potential of its divergence in some region near the support. Indeed, it follows from Maxwell's equations for the magnetostatic case [5, Ch. 5] that the magnetization distribution and its associated magnetic field are connected through an elliptic partial differential equation of Poisson type. More precisely, the magnetic field is the gradient of a scalar magnetic potential whose Laplacian is the divergence of the magnetization.

The main feature of this inverse problem is the geometry of the measurement set. In this work, we consider the case where measurements are taken on a plane which does not intersect the support of the magnetization. For instance, this setup is typical of experiments conducted with scanning superconducting quantum interference device (SQUID) microscopes when studying rock samples, see [10, 6]. In practice, two additional constraints complicate the situation further. The first is that only a single component of the field can be measured, namely the one that is orthogonal to the measurement plane: owing to the proximity to the sample and the very high sensitivity required, it is very difficult to accommodate more than one SQUID sensor without compromising accuracy, magnetic moment sensitivity, and spatial resolution. The second constraint is that measurements can only be performed on a finite portion of the measurement plane, close enough to the sample.

In analogy with the expansion of a dipolar field at infinity, the first and most basic issue comprises obtaining formulas connecting the normal component of the magnetic field on a plane to the integral of the magnetization (*i.e.*, net moment) that generates it, in such a way that knowledge of this component on part of that plane yields approximate formulas for the moment. Surprisingly, no such formulas seem readily available in the literature and the goal of the present paper is to provide useful expressions for estimating the net moment of compactly supported magnetizations from measurements of the normal component of its associated magnetic field taken on a portion of a plane.

We point out that net moment estimation is part of the larger inverse problem of full magnetization recovery from magnetic field measurements. The latter is ill-posed, not even injective. For thin supports (that can be identified with planar sets), non-uniqueness issues are analyzed in [2] and some recovery schemes are considered in [7] for unidirectional magnetizations. In contrast, net moment recovery is well-posed in that magnetizations producing the same normal component for the magnetic field on an open set of a plane must have the same moment (see [1]). Besides, the net moment provides valuable information on the magnetization itself, which may be used in full recovery schemes. This further motivates investigating recovery schemes for the moment of the magnetization.

Up to a rotation, we may assume for the ease of discussion that the measurement plane is horizontal and that the magnetization distribution is located below this plane. The questions we face are thus: (i) how can the vertical component of the magnetic field on a portion of the horizontal plane be used to yield an approximation of the moment of the magnetization generating that field? (ii) How does the error decay when that portion of horizontal plane grows large? In this regard, we mention that such asymptotics for the net moment were taken up in [9, Part III, Sec. 5, 6] for circular measurement areas using Fourier techniques and tools from harmonic analysis. As we will see below, formulas of a similar type can be obtained using elementary properties of homogeneous polynomials and Taylor expansions for rectangular measurement sets.

In the present work, we carry out in detail the corresponding computations when the measurement set is a square. The overview is as follows. The problem is set up in Section 2 and the main approximation results are stated in Section 3. Their proofs are given in Sections 4, 5 and in the Appendix. We discuss in Section 6 how these results can be used to provide estimates of the net moment of a magnetization when measurements of the field B_3 it generates are available. Finally, Section 7 contains concluding remarks.

2 Notation and problem setting

Given $s\,,\ r > 0$, we consider a parallelepiped $\mathcal{A} = [-s, s]^2 \times [0, r] \subset \mathbb{R}^3$ to contain the volume of the sample. Arbitrary points of $\mathbb{R}^3$ will be denoted as $\vec{x} = (x_1, x_2, x_3)$, while $\vec{t} = (t_1, t_2, t_3)$ will represent an arbitrary point of $\mathcal{A}$. For $i = 1, 2, 3$, we suppose we are given a real-valued function $m_i \in L^1(\mathcal{A})$, the Lebesgue space of summable functions on $\mathcal{A}$. We denote by $\vec{m}$ the magnetization vector field (m_1, m_2, m_3) of components m_i. A volumetric magnetization compactly supported on the slab $\mathcal{A}$ is modeled by the vector field on $\mathbb{R}^3$, $\vec{x} \mapsto \widetilde{\vec{m}}(\vec{x})$ where, for $i = 1, 2, 3$, $\widetilde{m_i}$ denotes the function m_i extended by 0 outside $\mathcal{A}$, *i.e.*, $\widetilde{m_i}(\vec{x}) = m_i(\vec{x})$ if $\vec{x} \in \mathcal{A}$ and $\widetilde{m_i}(\vec{x}) = 0$ otherwise. For any $m \in L^1(\mathcal{A})$, we denote by $\langle m \rangle$ the net moment given by the mean value of m:

$$\langle m \rangle = \iiint_{\mathcal{A}} m(\vec{t}) \, \mathrm{d}\vec{t}.$$

More generally, the net moment of m is its 0-th order moment, while the 1-st order moments are the quantities $\langle t_1 \, m \rangle$, $\langle t_2 \, m \rangle$, and $\langle t_3 \, m \rangle$, the 2-nd order moments are the quantities $\langle t_i \, t_j \, m \rangle$ (with arbitrary i and j in $\{1, 2, 3\}$), etc.

As recalled in [2], the magnetic field produced by the magnetized slab $(\mathcal{A}, \vec{m})$ is $\mathbf{B} = -\mu_0 \nabla \phi$,

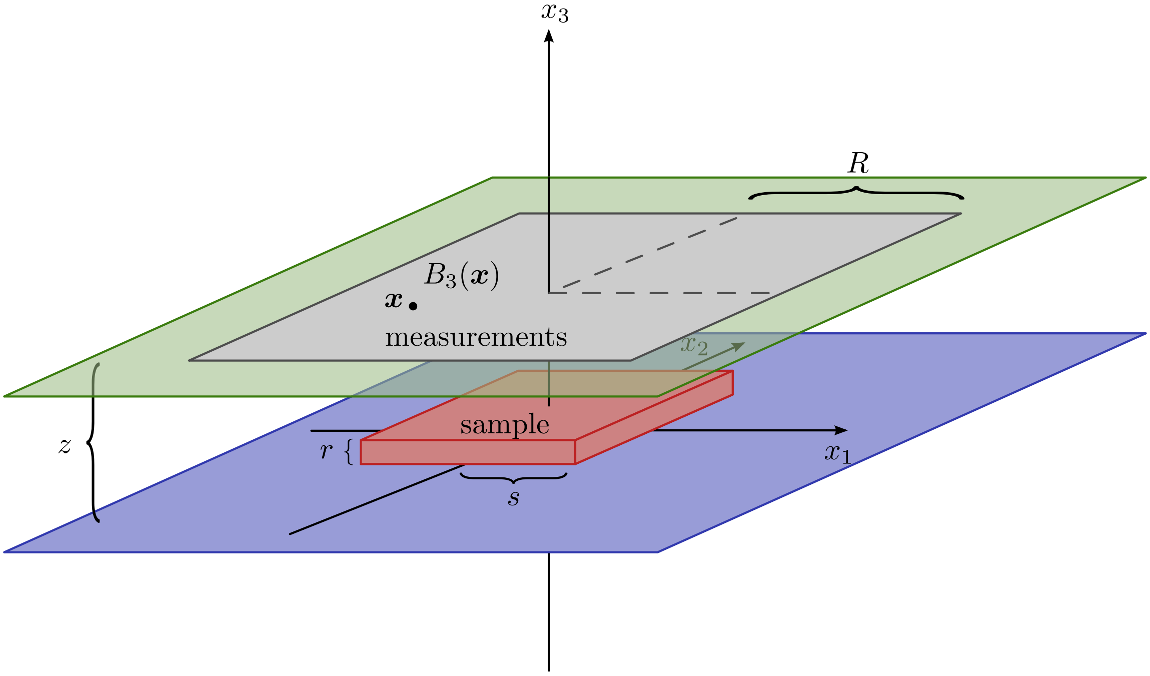

Figure 1: General view of the framework, with main notations.

where $\mu_0 = 4\pi \times 10^{-7}$ and ϕ is the scalar magnetic potential defined at each point $\vec{x} \notin \mathcal{A}$ by

$$\phi(\vec{x}) = \frac{1}{4\pi} \iiint_{\mathcal{A}} \frac{\langle \vec{m}(\vec{t}), \vec{x} - \vec{t} \rangle}{\|\vec{x} - \vec{t}\|^3} \, \mathrm{d}\vec{t}, \tag{1}$$

with $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ to designate the Euclidean norm and $\langle \vec{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$ the Euclidean scalar product. Denoting by P_i the function $\vec{x} \mapsto \frac{x_i}{\|\vec{x}\|^3}$, for $i = 1, 2, 3$, and by $\star$ the convolution product between functions defined on $\mathbb{R}^3$, we can express Equation (1) as the following shorter expression:

$$\phi(\vec{x}) = \frac{1}{4\pi} \left(P_1 \star \widetilde{m}_1 + P_2 \star \widetilde{m}_2 + P_3 \star \widetilde{m}_3 \right) (\vec{x}). \tag{2}$$

In the following, we will assume that we have measurements of the vertical component B_3 of $\mathbf{B}$ at a given height $x_3 = z > r$. This defines a function on the plane and we denote by $B_3[\vec{m}, z](x_1, x_2) = B_3(x_1, x_2, z)$ its value at a point $(x_1, x_2) \in \mathbb{R}^2$. For $\vec{x} \notin \mathcal{A}$, we have:

$$-\frac{4\pi}{\mu_0} B_3[\vec{m}, x_3](x_1, x_2) \quad = \quad \partial_{x_1} (P_3 \star \widetilde{m}_1 - P_1 \star \widetilde{m}_3)(\vec{x}) + \ \partial_{x_2} (P_3 \star \widetilde{m}_2 - P_2 \star \widetilde{m}_3)(\vec{x})\,. \tag{3}$$

This is easily checked by a direct computation: on the one hand, starting from $B_3[\vec{m}, x_3](x_1, x_2) = -\mu_0 \partial_{x_3} \phi(\vec{x})$ and computing the derivative with respect to x_3 of ϕ as given in Equation (1); on the other hand, explicitly computing the derivatives with respect to x_1 and x_2 in Equation (3). Another (deeper but more involved) way of seeing it consists in recognizing Poisson and Riesz transforms in Equation (2) and using their properties with respect to differentiation (see [1] for instance, where this is done in the case of a 2D slab).

Finally, for $R > 0$, we introduce the planar measurement areas $Q_R = [-R, R]^2$ (square), $S_R = \{(x_1, x_2) \in \mathbb{R}^2, |x_1| + |x_2| \leq R\}$ (diamond) and $A_R = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq R^2\}$ (disk) as illustrated in Figure 2. The symmetry of these measurement areas with respect to the origin is of essential use in the computations.

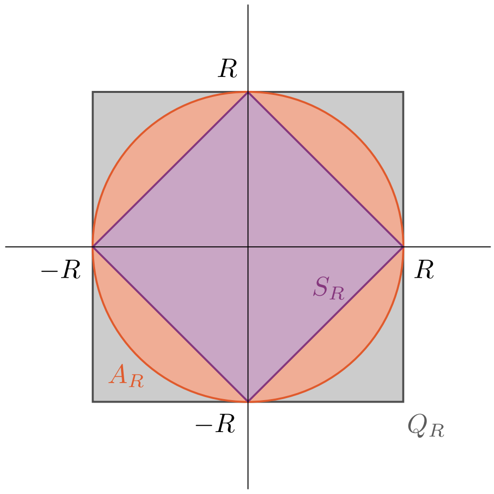

Figure 2: Shapes of Q_R, S_R, A_R.

3 Main results

Our main result is summed up in the following theorem that provides asymptotic expansions (as R goes large) of simple integrals involving $B_3[\vec{m}, z]$, in terms of the successive moments of the magnetization $\vec{m}$.

Theorem 1. *Let notation and assumptions be as in Section 2. On the square Q_R, it holds that:*

$$\iint_{Q_R} x_1\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 = \frac{\mu_0}{2}\, \langle m_1 \rangle + \frac{3\mu_0}{\pi R\sqrt{2}}\, (\langle t_1\, m_3 \rangle + \langle t_3\, m_1 \rangle - z\langle m_1 \rangle) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{4}$$

$$\iint_{Q_R} x_2\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 = \frac{\mu_0}{2}\, \langle m_2 \rangle + \frac{3\mu_0}{\pi R\sqrt{2}}\, (\langle t_2\, m_3 \rangle + \langle t_3\, m_2 \rangle - z\langle m_2 \rangle) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{5}$$

$$\begin{aligned}\iint_{Q_R} R\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 &= \frac{2\mu_0}{\pi\sqrt{2}} \langle m_3 \rangle \\ &+ \frac{5\mu_0}{4\pi R^2\sqrt{2}} \Big(- 2z^2 \langle m_3 \rangle + 2z\, (2\langle t_3\, m_3 \rangle - \langle t_1\, m_1 \rangle - \langle t_2\, m_2 \rangle) \\ &\quad + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle + 2\langle t_1 t_3\, m_1 \rangle + 2\langle t_2 t_3\, m_2 \rangle \Big) + \mathcal{O}\Big(\frac{1}{R^3}\Big),\end{aligned} \tag{6}$$

On the diamond S_R, it holds that:

$$\iint_{S_R} x_1\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 = \frac{\mu_0}{2}\, \langle m_1 \rangle + \frac{3\mu_0}{\pi R}\, (\langle t_1\, m_3 \rangle + \langle t_3\, m_1 \rangle - z\langle m_1 \rangle) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{7}$$

$$\iint_{S_R} x_2\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 = \frac{\mu_0}{2}\, \langle m_2 \rangle + \frac{3\mu_0}{\pi R}\, (\langle t_2\, m_3 \rangle + \langle t_3\, m_2 \rangle - z\langle m_2 \rangle) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{8}$$

$$\begin{aligned}\iint_{S_R} R\, B_3[\vec{m}, z](x_1, x_2)\, dx_1\, dx_2 &= \frac{2\mu_0}{\pi} \langle m_3 \rangle \\ &+ \frac{5\mu_0}{2\pi R^2} \Big(- 2z^2 \langle m_3 \rangle + 2z\, (2\langle t_3\, m_3 \rangle - \langle t_1\, m_1 \rangle - \langle t_2\, m_2 \rangle) \\ &\quad + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle + 2\langle t_1 t_3\, m_1 \rangle + 2\langle t_2 t_3\, m_2 \rangle \Big) + \mathcal{O}\Big(\frac{1}{R^3}\Big).\end{aligned} \tag{9}$$

Remark: we do not prove it in this article (for the sake of keeping computations reasonably simple) but the $\mathcal{O}(1/R^3)$ term of Equations (6) and (9) is actually a $\mathcal{O}(1/R^4)$.

The forthcoming Section 4 will establish Equations (4) and (6). We shall deduce all the other equations from the latter by means of appropriate changes of variable. This is performed in the dedicated Section 5. We conclude the present section with some comments about Theorem 1.

Thin slab hypothesis. In some contexts, the thickness of the magnetization slab is much smaller than its distance to the measurement plane, such that, from a practical point of view, one can consider its height r as being 0, *i.e.*, the slab is embedded in a plane instead of a volume. The magnetization is then modeled as a planar magnetization distribution [2]. If m denotes a component of such a magnetization (which therefore depends only on variable t_1 and t_2 since in this case $\mathcal{A}$ is simply the square $[-s,s]^2$), its net moment is then given by $\langle m \rangle = \iint_{\mathcal{A}} m(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2$. Higher order moments are defined accordingly, and when they involve t_3 they are simply equal to 0. In practice, asymptotic formulas for this thin-slab context are hence simply the same as those given in Theorem 1, when ignoring the higher-order moments containing t_3.

Adjoint operator. Notice that, if u is a function in $L^1(\mathbb{R}^2)$, we have (for $x_3 > r$)

$$\iint_{\mathbb{R}^2} u(x_1,x_2) B_3[\vec{m},x_3](x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = \iiint_{\mathcal{A}} m_1(\vec{t})\,v_1(\vec{t}) + m_2(\vec{t})\,v_2(\vec{t}) + m_3(\vec{t})\,v_3(\vec{t})\,\mathrm{d}\vec{t}$$

where $\vec{v}(\vec{t}) = (v_1(\vec{t}), v_2(\vec{t}), v_3(\vec{t}))$ is given by

$$\vec{v}(\vec{t}) = \frac{\mu_0}{4\pi}\,\partial_{x_3} \iint_{\mathbb{R}^2} \frac{\vec{t}-\vec{x}}{\|\vec{t}-\vec{x}\|^3}\,u(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2. \tag{10}$$

This simply follows from the definition of $\mathbf{B}$ from ϕ and from Equation (1), using Fubini's theorem and the fact the derivation with respect to x_3 commutes with all involved integrals. One can also easily check (explicitly computing the derivatives) that

$$\vec{v}(\vec{t}) = -\frac{\mu_0}{4\pi}\,\nabla\left((t_3 - x_3)\iint_{\mathbb{R}^2} \frac{u(x_1,x_2)}{\|\vec{t}-\vec{x}\|^3}\,\mathrm{d}x_1\mathrm{d}x_2\right) \tag{11}$$

where ∇ denotes the gradient: $\nabla = (\partial_{t_1}, \partial_{t_2}, \partial_{t_3})$. Equations (10) and (11) are reminiscent of formulas of the adjoint operator presented in [1] in the case of magnetizations in $L^2(\mathcal{A})$ where $\mathcal{A}$ is a two-dimensional thin slab.

Our proof of Equations (4) and (6) essentially consists in taking $u(x_1,x_2) = x_1\chi_{Q_R}$ and $u(x_1,x_2) = R\chi_{Q_R}$ where χ_{Q_R} denotes the characteristic function of the square Q_R, and doing an asymptotic expansion of the corresponding function $\vec{v}(\vec{t})$ with respect to R (and with x_3 being fixed to z). It turns out that Equation (11) can be explicitly computed in these cases (thanks to functions k and ℓ introduced in Definition 1, see Proposition 1 and following in the next section). The main issue is then to ensure that the remainder of the asymptotic expansion commutes with the integration on $\vec{t} \in \mathcal{A}$, which we achieve by computing the explicit dependence of the remainder with respect to variable $\vec{t}$. As will appear in the next section, we actually proceed slightly differently mainly to keep computations reasonably simple (the explicit expression of $\vec{v}(\vec{t})$ obtained from Equation (11) is quite large, with many similar terms, so we believe it is easier to conduct the computations in a more *ad hoc* way), but the overall strategy follows the same idea.

Magnetic moment magnetometer. Observe that, at first order, the integrals of Theorem 1 provide us with estimates for the quantities $\langle m_i \rangle$ $(i = 1,2,3)$ and can hence be used to implement a *moment magnetometer*. This is in fact what motivated this work and this idea will be developed further in Section 6, where we combine the expansions on the square Q_R and the diamond S_R to eliminate the term in $1/R$ in the expansions and achieve a better accuracy. Also,we point out the fact that, even though the remainders are given in Theorem 1 in the $\mathcal{O}(\cdot)$ form (which is the usual notation for truncated asymptotic expansions) our proof actually computes bounds that are explicit with respect to the characteristic quantities s, z and R (namely the bounds of function δ_{18}, resp. δ_{23}, at the end of Section 4.2, resp. 4.3).

Disk geometry. Similar asymptotic formulas can be obtained for the disk A_R centered at 0 and of radius R, namely it holds that:

$$\iint_{A_R} x_1\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mu_0}{2} \langle m_1 \rangle + \frac{3\mu_0}{4R} \left(\langle t_1\, m_3 \rangle + \langle t_3\, m_1 \rangle - z \langle m_1 \rangle \right) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{12}$$

$$\iint_{A_R} x_2\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mu_0}{2} \langle m_2 \rangle + \frac{3\mu_0}{4R} \left(\langle t_2\, m_3 \rangle + \langle t_3\, m_2 \rangle - z \langle m_2 \rangle \right) + \mathcal{O}\Big(\frac{1}{R^3}\Big), \tag{13}$$

$$\begin{aligned} \iint_{A_R} R\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mu_0}{2} \langle m_3 \rangle \\ + \frac{3\mu_0}{8R^2} \Big(- 2z^2 \langle m_3 \rangle + 2z \left(2 \langle t_3\, m_3 \rangle - \langle t_1\, m_1 \rangle - \langle t_2\, m_2 \rangle \right) \\ + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2 \langle t_3^2 m_3 \rangle + 2 \langle t_1 t_3\, m_1 \rangle + 2 \langle t_2 t_3\, m_2 \rangle \Big) + \mathcal{O}\Big(\frac{1}{R^4}\Big). \end{aligned} \tag{14}$$

We will not prove these formulas. They can be obtained from [9, Part III, Sec. 3.5, (3.75)] and the related formulas in [9, Part III, Sec. 3.6], taking into account the 3-D character of the slab $\mathcal{A}$.

4 Proofs of Equations (4) and (6)

4.1 Preliminary results

Before going to the actual proof of Equations (4) and (6) we define functions and establish results that will be of constant use in what follows.

Definition 1. *We define the $\mathbb{R}$-valued functions f, g, k and ℓ for $\vec{x} \in \mathbb{R}^3$ with $x_3 > 0$ by:*

$$\begin{aligned} f(\vec{x}) &= \frac{x_2}{x_1^2 + x_3^2} \cdot \frac{1}{\|\vec{x}\|}, \\ g(\vec{x}) &= \frac{-1}{\|\vec{x}\|}, \\ k(\vec{x}) &= \frac{1}{x_3} \arctan\left(\frac{x_1\, x_2}{x_3\, \|\vec{x}\|} \right), \\ \ell(\vec{x}) &= -\operatorname{arcsinh}\left(\frac{x_2}{(x_1^2 + x_3^2)^{1/2}} \right). \end{aligned}$$

These functions are indefinite integrals of expressions that will naturally come up when rewriting the left hand sides of Equations (4) and (6). This is capsulized in the following proposition whose proof reduces to straightforward computations. Below, the symbol ∂_{x_i} stands for the partial derivative with respect to the coordinate x_i ($i = 1, 2, 3$) while $\partial^2_{x_1 x_2} = \partial_{x_1} \partial_{x_2}$.

Proposition 1. *For any $\vec{x} \in \mathbb{R}^3$ with $x_3 > 0$, we have*

$$\begin{array}{rcl} \partial_{x_2} f(\vec{x}) & = & 1/\|\vec{x}\|^3 \,, \\ \partial_{x_1} g(\vec{x}) & = & x_1/\|\vec{x}\|^3 \,, \\ \partial^2_{x_1 x_2} k(\vec{x}) & = & 1/\|\vec{x}\|^3 \,, \\ \partial^2_{x_1 x_2} \ell(\vec{x}) & = & x_1/\|\vec{x}\|^3 \,. \end{array}$$

We will need asymptotic expansions of expressions of the form $f(R-t_1, R-t_2, z-t_3)$, $f(-R-t_1, R-t_2, z-t_3)$, etc., when R goes large. To this effect, it is convenient to introduce companion functions to f, g, k and ℓ as follows.

Definition 2. *Let $\vec{t} = (t_1, t_2, t_3) \in \mathcal{A}$. We define $F_{\vec{t}}$, $G_{\vec{t}}$, $K_{\vec{t}}$ and $L_{\vec{t}}$ from $\mathbb{R}$ to $\mathbb{R}$ by*

$$\begin{array}{ll} F_{\vec{t}}(R) = f(R-t_1, R-t_2, z-t_3), & G_{\vec{t}}(R) = g(R-t_1, R-t_2, z-t_3), \\ K_{\vec{t}}(R) = k(R-t_1, R-t_2, z-t_3), \quad \textit{and} & L_{\vec{t}}(R) = \ell(R-t_1, R-t_2, z-t_3). \end{array}$$

One easily checks the following proposition.

Proposition 2. *For any $(t_1, t_2) \in [-s, s]^2$, any $t_3 > 0$ and any $R \in \mathbb{R}$,*

$$\begin{array}{ll} f(-R-t_1, R-t_2, z-t_3) = F_{(-t_1,t_2,t_3)}(R), & g(-R-t_1, R-t_2, z-t_3) = G_{(-t_1,t_2,t_3)}(R), \\ f(-R-t_1, -R-t_2, z-t_3) = -F_{(-t_1,-t_2,t_3)}(R), & g(-R-t_1, -R-t_2, z-t_3) = G_{(-t_1,-t_2,t_3}(R), \\ f(R-t_1, -R-t_2, z-t_3) = -F_{(t_1,-t_2,t_3)}(R); & g(R-t_1, -R-t_2, z-t_3) = G_{(t_1,-t_2,t_3)}(R); \\ k(-R-t_1, R-t_2, z-t_3) = -K_{(-t_1,t_2,t_3)}(R), & \ell(-R-t_1, R-t_2, z-t_3) = L_{(-t_1,t_2,t_3)}(R), \\ k(-R-t_1, -R-t_2, z-t_3) = K_{(-t_1,-t_2,t_3)}(R), & \ell(-R-t_1, -R-t_2, z-t_3) = -L_{(-t_1,-t_2,t_3)}(R), \\ k(R-t_1, -R-t_2, z-t_3) = -K_{(t_1,-t_2,t_3)}(R); & \ell(R-t_1, -R-t_2, z-t_3) = -L_{(t_1,-t_2,t_3)}(R). \end{array}$$

The essential ingredient for the proof of Equations (4) and (6) is to get asymptotic expansions of the functions $F_{\vec{t}}$, $G_{\vec{t}}$, $K_{\vec{t}}$, $L_{\vec{t}}$ (with respect to powers of $1/R$), with explicit error bounds. The important point is that these error bounds are uniform with respect to the variable $\vec{t} \in \mathcal{A}$, allowing us to integrate them on $\mathcal{A}$. Such expansions are given in Lemma 1 below. Before stating it, we need to introduce more notations and to recall some properties of homogeneous polynomials.

We fix once and for all two positive constants ω_s and ω_z such that $\omega_s < 1$, and we pose

$$C = \max\left\{\frac{s}{\omega_s}, \frac{z}{\omega_z}\right\}, \tag{15}$$

hence $\frac{s}{C} \leq \omega_s < 1$ and $\frac{z}{C} \leq \omega_z$. From now on, we assume that $R \geq C$ with C given by Equation (15).

Remark 1. Introducing the rescaling quantities ω_s and ω_z is a means to assume that $R \geq C$ whatever the dimensions s and r of $\mathcal{A}$ and the height z of the measurement area (on can simply take $\omega_z = z/R$ and $\omega_s = s/R$, if $R > s$). The assumption $R \geq C$ reflects of course the asymptotic character of the present study. Moreover, the quantities ω_s and ω_z conveniently allow us to specify how large R should be relative to s and z for the error estimates in the expansions below to hold true (see Lemmas to come and Remark 2 at the end of the Appendix).

Let $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$. If $a_n(\vec{\gamma})$ is a homogeneous polynomial of degree n with real coefficients, *i.e.*,

$$a_n(\vec{\gamma}) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \alpha_{i,j}\, \gamma_1^i\, \gamma_2^j\, \gamma_3^{n-i-j}\,,$$

where $\alpha_{i,j} \in \mathbb{R}$, we define the associated homogeneous polynomial $A_n(s,z)$ in the variables s and z by:

$$A_n(s,z) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} |\alpha_{i,j}| \; s^{i+j} \; z^{n-(i+j)}. \tag{16}$$

Observe that whenever $\vec{\gamma} \in [-s,s]^2 \times (0,z]$, we have $|a_n(\vec{\gamma})| \le A_n(s,z)$. In particular (recall from Section 2 that $\mathcal{A} = [-s,s]^2 \times [0,r]$ and $z > r$),

$$\forall \vec{t} \in \mathcal{A}, \quad |a_n(t_1, t_2, z - t_3)| \le A_n(s,z). \tag{17}$$

Moreover, if

$$\Delta_n(s,z) = \sum_{k=0}^{n} b_k \; s^k \; z^{n-k}$$

is a homogeneous polynomial of degree n in s and z with $b_k \in \mathbb{R}$, we define another associated homogeneous polynomial $\Delta_{n,1}(s,z)$ of degree $n-1$ by

$$\Delta_{n,1}(s,z) = (|b_0|\omega_z) z^{n-1} + \sum_{k=1}^{n} (|b_k|\omega_s) \; s^{k-1} z^{n-k}. \tag{18}$$

Notice that for any $\xi \in [0, 1/C]$ we have $|z\xi| \le \omega_z$ and $|s\xi| \le \omega_s$ by Equation (15); therefore $|\Delta_n(s,z)\; \xi^n| \le \Delta_{n,1}(s,z)\; \xi^{n-1}$. We denote by $\Delta_{n,2}(s,z)$ the polynomial obtained by applying the same process to $\Delta_{n,1}(s,z)$, *i.e.*, $\Delta_{n,2}(s,z) = \Delta_{n,1,1}(s,z)$, and more generally we put $\Delta_{n,p+1}(s,z) = \Delta_{n,p,1}(s,z)$. By simple induction, we see that $\Delta_{n,p}(s,z)$ is a homogeneous polynomial of degree $n-p$ and

$$\forall \xi \in [0, 1/C], \quad |\Delta_n(s,z)\; \xi^n| \le \Delta_{n,p}(s,z)\; \xi^{n-p}. \tag{19}$$

As a particular case, observe that $\Delta_{n,n}(s,z)$ is in fact a constant and satifies $|\Delta_n(s,z)\xi^n| \le \Delta_{n,n}$.

We can now state our first lemma. We postpone its proof to the appendix at the end of this document.

Lemma 1. *For any $\vec{t} \in \mathcal{A}$ and any $R \ge C$, where C is defined by Equation (15), it holds*

$$F_{\vec{t}}(R) = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10t_3^2 + 20zt_3 - 10z^2}{8\sqrt{2}} \cdot \frac{1}{R^4} + \frac{\delta_1(\vec{t}, z, R)}{R^5}$$

$$G_{\vec{t}}(R) = \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} - \frac{t_1^2 + t_2^2 + 6t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{8\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_2(\vec{t}, z, R)}{R^4}$$

$$K_{\vec{t}}(R) = \frac{\pi}{2(z - t_3)} - \sqrt{2}\,\frac{1}{R} - \frac{\sqrt{2}(t_1 + t_2)}{2} \cdot \frac{1}{R^2} + \frac{\delta_3(\vec{t}, z, R)}{R^3},$$

$$L_{\vec{t}}(R) = -\operatorname{arcsinh}(1) + \frac{t_2 - t_1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{3t_1^2 - t_2^2 - 2t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{4\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_4(\vec{t}, z, R)}{R^3},$$

where $|\delta_1(\vec{t}, z, R)| \le \Delta_3^{(1)}(s,z)$, $|\delta_2(\vec{t}, z, R)| \le \Delta_3^{(2)}(s,z)$, $|\delta_3(\vec{t}, z, R)| \le \Delta_2^{(3)}(s,z)$ and $|\delta_4(\vec{t}, z, R)| \le \Delta_3^{(4)}(s,z)$ for some homogeneous polynomials $\Delta_n^{(i)}$ in the variables s and z, of degree n.

Corollary 1. *Under the same hypotheses as in Lemma 1, it holds*

$$F_{\vec{t}}(R) = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_5(\vec{t}, z, R)}{R^4},$$

$$G_{\vec{t}}(R) = \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_6(\vec{t}, z, R)}{R^3},$$

where $|\delta_5(\vec{t}, z, R)| \le \Delta_2^{(5)}(s,z)$ *and* $|\delta_6(\vec{t}, z, R)| \le \Delta_2^{(6)}(s,z)$, *for some homogeneous polynomials* $\Delta_2^{(5)}$ *and* $\Delta_2^{(6)}$ *in the variables* s *and* z, *of degree* 2.

Proof. From Lemma 1 we get

$$|\delta_5(\vec{t}, z, R)| = \left| \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10(z-t_3)^2}{8\sqrt{2}} + \frac{\delta_1(\vec{t}, z, R)}{R} \right| \le \frac{42s^2 + 10z^2}{8\sqrt{2}} + \frac{\Delta_3^{(1)}(s,z)}{R}.$$

Now, since $1/R \in [0, 1/C]$, Equation (19) implies that $|\Delta_3^{(1)}(s,z) \cdot 1/R| \le \Delta_{3,1}^{(1)}(s,z)$. Consequently, $|\delta_5(\vec{t}, z, R)| \le \Delta_2^{(5)}(s,z)$ where $\Delta_2^{(5)}(s,z) = \frac{42s^2+10z^2}{8\sqrt{2}} + \Delta_{3,1}^{(1)}(s,z)$ is a homogeneous polynomial of degree 2. The result for $G_{\vec{t}}(R)$ is obtained similarly. $\square$

4.2 Proof of Equation (4)

From Equation (3), we get on integrating the term $x_1 \partial_{x_1}$by parts that :

$$\begin{aligned}
-\frac{4\pi}{\mu_0} \iint_{Q_R} x_1\, B_3[\vec{m}, x_3](x_1,x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = & \int_{-R}^{R} \Big[x_1 \left(P_3 \star \widetilde{m}_1 - P_1 \star \widetilde{m}_3 \right)(\vec{x}) \Big]_{x_1=-R}^{R} \mathrm{d}x_2 \\
& + \int_{-R}^{R} \Big[x_1 \left(P_3 \star \widetilde{m}_2 - P_2 \star \widetilde{m}_3 \right)(\vec{x}) \Big]_{x_2=-R}^{R} \mathrm{d}x_1 \\
& - \iint_{Q_R} P_3 \star \widetilde{m}_1(\vec{x})\, \mathrm{d}x_1 \mathrm{d}x_2 \\
& + \iint_{Q_R} P_1 \star \widetilde{m}_3(\vec{x})\, \mathrm{d}x_1 \mathrm{d}x_2.
\end{aligned}$$

Now, replacing P_1, P_2 and P_3 by their expressions and using Fubini's theorem to interchange the integration on Q_R and the integration on $\mathcal{A}$ arising from the convolution, we get in view of Proposition 1 that

$$\iint_{Q_R} x_1\, B_3[\vec{m}, x_3](x_1,x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = -\frac{\mu_0}{4\,\pi} \iiint_{\mathcal{A}} \left(I_1(\vec{t}) + I_2(\vec{t}) + I_3(\vec{t}) + I_4(\vec{t}) \right) \mathrm{d}\vec{t}, \tag{20}$$

where, for $\vec{t} \in \mathcal{A}$:

$$\begin{aligned}
I_1(\vec{t}) &= \Big[\Big[\left(x_1(x_3-t_3)\, m_1(\vec{t}) - x_1(x_1-t_1)\, m_3(\vec{t}) \right) f(\vec{x}-\vec{t}) \Big]_{x_1=-R}^{R} \Big]_{x_2=-R}^{R}, \\
I_2(\vec{t}) &= \Big[\left((x_3-t_3)\, m_2(\vec{t}) - (x_2-t_2)\, m_3(\vec{t}) \right) \int_{-R}^{R} \frac{x_1}{\|\vec{x}-\vec{t}\|^3} \mathrm{d}x_1 \Big]_{x_2=-R}^{R}, \\
I_3(\vec{t}) &= -(x_3-t_3)\, m_1(\vec{t}) \Big[\Big[k(\vec{x}-\vec{t}) \Big]_{x_1=-R}^{R} \Big]_{x_2=-R}^{R}, \\
I_4(\vec{t}) &= m_3(\vec{t}) \Big[\Big[\ell(\vec{x}-\vec{t}) \Big]_{x_1=-R}^{R} \Big]_{x_2=-R}^{R}.
\end{aligned}$$

To simplify $I_2(\vec{t})$ further, we can rewrite x_1 as $(x_1-t_1)+t_1$ and $\|\vec{x}-\vec{t}\|$ as $\|(x_2-t_2, x_1-t_1, x_3-t_3)\|$, which leads us to

$$\int_{-R}^{R} \frac{x_1}{\|\vec{x}-\vec{t}\|^3} \mathrm{d}x_1 = \Big[g(\vec{x}-\vec{t}) + t_1 f(x_2-t_2, x_1-t_1, x_3-t_3) \Big]_{x_1=-R}^{R}.$$

Now, grouping terms according to the parity of the powers of x_1 and x_2, we obtain the following expressions for $I_1(\vec{t})$ and $I_2(\vec{t})$:

$$\begin{aligned}
I_1(\vec{t}) &= ((x_3-t_3)\, m_1(\vec{t}) + t_1\, m_3(\vec{t}))\, \Big[\Big[x_1 f(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} - m_3(\vec{t})\, \Big[\Big[x_1^2\, f(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R}, \\
I_2(\vec{t}) &= ((x_3-t_3)\, m_2(\vec{t}) + t_2\, m_3(\vec{t}))\, \Big[\Big[g(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&\quad + t_1((x_3-t_3)\, m_2(\vec{t}) + t_2\, m_3(\vec{t}))\, \Big[\Big[f(x_2-t_2, x_1-t_1, x_3-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&\quad -\; m_3(\vec{t})\Big[\Big[x_2 g(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&\quad - t_1 m_3\, \Big[\Big[x_2 f(x_2-t_2, x_1-t_1, x_3-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R}.
\end{aligned}$$

From now on, we assume that x_3 is fixed and equal to z and that the hypotheses of Lemma 1 are satisfied.

Asymptotic expansion of I_1. Using Proposition 2 and Lemma 1, we get

$$\begin{aligned}
\Big[\Big[x_1 f(\vec{x}-\vec{t})_{|x_3=z}\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= R\, \Big(F_{\vec{t}}(R) + F_{(-t_1,t_2,t_3)}(R) + F_{(t_1,-t_2,t_3)}(R) + F_{(-t_1,-t_2,t_3)}(R)\Big) \\
&= \frac{4}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_7(\vec{t}, z, R)}{R^3}, \\
\Big[\Big[x_1^2\, f(\vec{x}-\vec{t})_{|x_3=z}\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= R^2\, \Big[\Big[f(\vec{x}-\vec{t})_{|x_3=z}\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&= R^2\, \Big(F_{\vec{t}}(R) - F_{(-t_1,t_2,t_3)}(R) + F_{(t_1,-t_2,t_3)}(R) - F_{(-t_1,-t_2,t_3)}(R)\Big) \\
&= \frac{10 t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_8(\vec{t}, z, R)}{R^3},
\end{aligned}$$

where $|\delta_7(\vec{t}, z, R)| \le 4\Delta_2^{(5)}(s, z)$ and $|\delta_8(\vec{t}, z, R)| \le 4\Delta_3^{(1)}(s, z)$. Therefore,

$$I_1(\vec{t}) = \frac{4(z-t_3)\, m_1(\vec{t}) - 6 t_1\, m_3(\vec{t})}{R\sqrt{2}} + \frac{\delta_9(\vec{t}, z, R)}{R^3} \tag{21}$$

where $|\delta_9(\vec{t}, z, R)| \le 4z|m_1(\vec{t})|\Delta_2^{(5)}(s, z) + 4|m_3(\vec{t})|(s\Delta_2^{(5)}(s, z) + \Delta_3^{(1)}(s, z))$.

Asymptotic expansion of I_2. A simple interchange between variables x_1 and x_2 and between variables t_1 and t_2 in the computations of the previous paragraph shows that

$$\begin{aligned}
\Big[\Big[f(x_2-t_2, x_1-t_1, z-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= \frac{10 t_2}{\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_8((t_2, t_1, t_3), z, R)}{R^5}, \\
&= \frac{\delta_{10}(\vec{t}, z, R)}{R^3}, \\
\Big[\Big[x_2 f(x_2-t_2, x_1-t_1, z-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= \frac{4}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_7((t_2, t_1, t_3), z, R)}{R^3},
\end{aligned}$$

where $|\delta_{10}(\vec{t}, z, R)| \le \frac{10s}{\sqrt{2}} + 4\Delta_3^{(1)}(s, z)\, \frac{1}{R^2}$. Now, observing that $1/R \in [0, 1/C]$, we obtain $|\delta_{10}(\vec{t}, z, R)| \le \Delta_1^{(10)}(s, z)$ where $\Delta_1^{(10)}(s, z)$ is the homogeneous polynomial of degree 1 defined by $\Delta_1^{(10)}(s,z) = \frac{10s}{\sqrt{2}} + 4\Delta_{3,2}^{(1)}(s,z)$. Here, $\Delta_{3,2}^{(1)}$ is the polynomial constructed from $\Delta_3^{(1)}$ by two successive applications of the process defined by Equation (18).

Moreover, using Proposition 2 and Lemma 1 on $g_{\vec{t}}$, we get

$$\begin{aligned}
\left[\left[g(\vec{x}-\vec{t})_{|x_3=z}\right]_{x_1=-R}^{R}\right]_{x_2=-R}^{R} &= G_{\vec{t}}(R) - G_{(-t_1,t_2,t_3)}(R) - G_{(t_1,-t_2,t_3)}(R) + G_{(-t_1,-t_2,t_3)}(R) \\
&= \frac{\delta_{11}(\vec{t},z,R)}{R^3}\,, \\
\left[\left[x_2 g(\vec{x}-\vec{t})_{|x_3=z}\right]_{x_1=-R}^{R}\right]_{x_2=-R}^{R} &= R\left(G_{\vec{t}}(R) - G_{(-t_1,t_2,t_3)}(R) + G_{(t_1,-t_2,t_3)}(R) - G_{(-t_1,-t_2,t_3)}(R)\right) \\
&= \frac{-2t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_{12}(\vec{t},z,R)}{R^3}\,,
\end{aligned}$$

where $|\delta_{11}(\vec{t},z,R)| \leq 4\Delta_2^{(6)}(s,z)$ and $|\delta_{12}(\vec{t},z,R)| \leq 4\Delta_3^{(2)}(s,z)$. Therefore,

$$I_2(\vec{t}) = -\frac{2t_1\, m_3(\vec{t})}{R\sqrt{2}} + \frac{\delta_{13}(\vec{t},z,R)}{R^3} \tag{22}$$

where

$$\begin{aligned}
|\delta_{13}(\vec{t},z,R)| \;\leq\; & (4\Delta_2^{(6)}(s,z) + s\Delta_1^{(10)}(s,z))\, z|m_2(\vec{t})| \\
& + (4s\Delta_2^{(6)}(s,z) + 4\Delta_3^{(2)}(s,z) + 4s\Delta_2^{(5)}(s,z) + s^2\Delta_1^{(10)}(s,z))\, m_3(\vec{t})
\end{aligned}$$

Asymptotic expansions of I_3 and I_4. Following the same line of argument as we used for I_1 and I_2, we get

$$\begin{aligned}
\left[\left[k(\vec{x}-\vec{t})_{|x_3=z}\right]_{x_1=-R}^{R}\right]_{x_2=-R}^{R} &= K_{\vec{t}}(R) + K_{(-t_1,t_2,t_3)}(R) + K_{(t_1,-t_2,t_3)}(R) + K_{(-t_1,-t_2,t_3)}(R) \\
&= \frac{2\pi}{z-t_3} - 4\sqrt{2}\frac{1}{R} + \frac{\delta_{14}(\vec{t},z,R)}{R^3}\,, \\
\left[\left[\ell(\vec{x}-\vec{t})_{|x_3=z}\right]_{x_1=-R}^{R}\right]_{x_2=-R}^{R} &= L_{\vec{t}}(R) - L_{(-t_1,t_2,t_3)}(R) + L_{(t_1,-t_2,t_3)}(R) - L_{(-t_1,-t_2,t_3)}(R) \\
&= \frac{-4t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_{15}(\vec{t},z,R)}{R^3}\,,
\end{aligned}$$

where $|\delta_{14}(\vec{t},z,R)| \leq 4\Delta_2^{(3)}(s,z)$ and $|\delta_{15}(\vec{t},z,R)| \leq 4\Delta_3^{(4)}(s,z)$. Therefore,

$$I_3(\vec{t}) = -2\pi m_1(\vec{t}) + \frac{4(z-t_3)\sqrt{2}\, m_1(\vec{t})}{R} + \frac{\delta_{16}(\vec{t},z,R)}{R^3}\,, \tag{23}$$

$$I_4(\vec{t}) = \frac{-4t_1 m_3(\vec{t})}{R\sqrt{2}} + \frac{\delta_{17}(\vec{t},z,R)}{R^3}, \tag{24}$$

where $|\delta_{16}(\vec{t},z,R)| \leq 4\Delta_2^{(3)}(s,z)z|m_1(\vec{t})|$ and $|\delta_{17}(\vec{t},z,R)| \leq 4\Delta_3^{(4)}(s,z)|m_3(\vec{t})|$.

Final step. Plugging Equations (21), (22), (23) and (24) into Equation (20), we finally get

$$\iint_{Q_R} x_1\, B_3[\vec{m},z](x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = \frac{\mu_0}{2}\,\langle m_1\rangle + \frac{3\mu_0}{\pi R\sqrt{2}}\,(\langle t_1\, m_3\rangle + \langle t_3\, m_1\rangle - z\langle m_1\rangle) + \frac{\delta_{18}(s,z,R)}{R^3},$$

where $\delta_{18}(s,z,R) = -\frac{\mu_0}{4\pi}\iiint_{\mathcal{A}}\Big(\delta_9(\vec{t},z,R)+\delta_{13}(\vec{t},z,R)+\delta_{16}(\vec{t},z,R)+\delta_{17}(\vec{t},z,R)\Big)\,\mathrm{d}\vec{t}$. From the inequalities obtained in the previous paragraphs, we see that, for any $R \geq C$, where C is the constant given by Equation (15),

$$
\begin{aligned}
|\delta_{18}(s,z,R)| \leq \frac{\mu_0}{4\pi}\Big(\, & 4(\Delta_2^{(5)}(s,z)+\Delta_2^{(3)}(s,z))\, z\langle|m_1|\rangle \\
& +(4\Delta_2^{(6)}(s,z)+s\Delta_1^{(10)}(s,z))\, z\langle|m_2|\rangle \\
& +(4(\Delta_3^{(1)}(s,z)+\Delta_3^{(2)}(s,z)+\Delta_3^{(4)}(s,z)) \\
& \quad +4s(2\Delta_2^{(5)}(s,z)+\Delta_2^{(6)}(s,z))+s^2\Delta_1^{(10)}(s,z))\,\langle|m_3|\rangle\Big)
\end{aligned}
$$

This shows that δ_{18} is bounded by a quantity that depends only on s and z but not on R.

4.3 Proof of Equation (6)

We follow the path that led us to Equation (4): starting from Equation (3) we get

$$
\begin{aligned}
-\frac{2}{\mu_0}\iint_{Q_R} R\,B_3[\vec{m},x_3](x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = & \int_{-R}^{R} R\Big[\,(P_3\star\widetilde{m}_1 - P_1\star\widetilde{m}_3)\,(\vec{x})\Big]_{x_1=-R}^{R}\mathrm{d}x_2 \\
& +\int_{-R}^{R} R\Big[\,(P_3\star\widetilde{m}_2 - P_2\star\widetilde{m}_3)\,(\vec{x})\Big]_{x_2=-R}^{R}\mathrm{d}x_1.
\end{aligned}
$$

Now, replacing P_1, P_2 and P_3 by their expressions and using Fubini, we obtain

$$
\iint_{Q_R} R\,B_3[\vec{m},x_3](x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\mu_0}{4\,\pi}\iiint_{\mathcal{A}}\Big(I_5(\vec{t})+I_6(\vec{t})\Big)\,\mathrm{d}\vec{t}, \tag{25}
$$

where:

$$
\begin{aligned}
I_5(\vec{t}) &= R\Big[\Big[((x_3-t_3)\,m_1(\vec{t})-(x_1-t_1)\,m_3(\vec{t}))\,f(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&= R((x_3-t_3)\,m_1(\vec{t})+t_1\,m_3(\vec{t}))\Big[\Big[f(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} - R\,m_3(\vec{t})\Big[\Big[x_1 f(\vec{x}-\vec{t})\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R},
\end{aligned}
$$

$$
\begin{aligned}
I_6(\vec{t}) &= R\Big[((x_3-t_3)\,m_2(\vec{t})-(x_2-t_2)\,m_3(\vec{t}))\,\int_{-R}^{R}\frac{1}{\|\vec{x}-\vec{t}\|}\,\mathrm{d}x_1\Big]_{x_2=-R}^{R} \\
&= R\Big[\Big[((x_3-t_3)\,m_2(\vec{t})-(x_2-t_2)\,m_3(\vec{t}))\,f(x_2-t_2,x_1-t_1,x_3-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&= R((x_3-t_3)\,m_2(\vec{t})+t_2\,m_3(\vec{t}))\Big[\Big[f(x_2-t_2,x_1-t_1,x_3-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} \\
&\quad -R\,m_3(\vec{t})\Big[\Big[x_2 f(x_2-t_2,x_1-t_1,x_3-t_3)\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R}.
\end{aligned}
$$

Now, using the same arguments as in Section 4.2, and assuming as before that x_3 is fixed and equal to z and that the hypotheses of Lemma 1 are satisfied, we see that

$$
\begin{aligned}
R\Big[\Big[f(\vec{x}-\vec{t})_{|x_3=z}\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= \frac{10t_1}{\sqrt{2}}\cdot\frac{1}{R^2}+\frac{\delta_{19}(\vec{t},z,R)}{R^3}, \\
R\Big[\Big[x_1 f(\vec{x}-\vec{t})_{|x_3=z}\Big]_{x_1=-R}^{R}\Big]_{x_2=-R}^{R} &= \frac{4}{\sqrt{2}}+\frac{33t_1^2-3t_2^2-10(z-t_3)^2}{2\sqrt{2}}\cdot\frac{1}{R^2}+\frac{\delta_{20}(\vec{t},z,R)}{R^3},
\end{aligned}
$$

where $|\delta_{19}(\vec{t},z,R)| \leq 4\Delta_3^{(1)}(s,z)\frac{1}{R}$ and $\delta_{20}(\vec{t},z,R) \leq 4\Delta_3^{(1)}(s,z)$. Therefore,

$$I_5(\vec{t}) = \frac{-4m_3(\vec{t})}{\sqrt{2}} + \frac{20(z-t_3)t_1 m_1(\vec{t}) - 13t_1^2 m_3(\vec{t}) + 3t_2^2 m_3(\vec{t}) + 10(z-t_3)^2 m_3(\vec{t})}{2R^2\sqrt{2}} + \frac{\delta_{21}(\vec{t},z,R)}{R^3},$$

where $|\delta_{21}(\vec{t},z,R)| \leq 4\Delta_3^{(1)}(s,z)\Big(|m_3(\vec{t})|(1+\frac{s}{R}) + |m_1(\vec{t})|\frac{z}{R}\Big)$ and therefore, when $R \geq C$, $|\delta_{21}(\vec{t},z,R)| \leq 4\Delta_3^{(1)}(s,z)\Big(|m_3(\vec{t})|(1+\omega_s) + |m_1(\vec{t})|\omega_z\Big)$ by Equation (15).

Accordingly,

$$I_6(\vec{t}) = \frac{-4m_3(\vec{t})}{\sqrt{2}} + \frac{20(z-t_3)t_2 m_2(\vec{t}) - 13t_2^2 m_3(\vec{t}) + 3t_1^2 m_3(\vec{t}) + 10(z-t_3)^2 m_3(\vec{t})}{2R^2\sqrt{2}} + \frac{\delta_{22}(\vec{t},z,R)}{R^3},$$

where $|\delta_{22}(\vec{t},z,R)| \leq 4\Delta_3^{(1)}(s,z)\Big(|m_3(\vec{t})|(1+\omega_s) + |m_2(\vec{t})|\omega_z\Big)$.

Putting these last two equations together we finally get

$$\begin{aligned}\iint_{Q_R} R\, B_3[\vec{m},z](x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 &= \frac{2\mu_0}{\pi\sqrt{2}}\langle m_3\rangle \\ &+ \frac{5\mu_0}{4\pi R^2\sqrt{2}}\Big(-2z^2\langle m_3\rangle + z\,(4\langle t_3\, m_3\rangle - 2\langle t_1\, m_1\rangle - 2\langle t_2\, m_2\rangle) \\ &\qquad + \langle t_1^2 m_3\rangle + \langle t_2^2 m_3\rangle - 2\langle t_3^2 m_3\rangle + 2\langle t_1 t_3\, m_1\rangle + 2\langle t_2 t_3\, m_2\rangle\Big) + \frac{\delta_{23}(s,z,R)}{R^3},\end{aligned}$$

where $\delta_{23}(s,z,R) = -\frac{\mu_0}{4\pi}\iiint_{\mathcal{A}}\Big(\delta_{21}(\vec{t},z,R) + \delta_{22}(\vec{t},z,R)\Big)\,\mathrm{d}\vec{t}$. Now, δ_{23} is bounded when R goes to infinity, since for any $R \geq C$ (where C is the constant given by Equation (15)) we have

$$|\delta_{23}(s,z,R)| \leq \frac{\mu_0}{\pi}\Delta_3^{(1)}(s,z)\Big(2\langle|m_3|\rangle(1+\omega_s) + \langle|m_1|\rangle\omega_z + \langle|m_2|\rangle\omega_z\Big).$$

5 Proofs of the remaining equations

5.1 Generalities

Consider a linear isometry Ψ of $\mathbb{R}^2$ and define the linear isometry $\bar{\Psi}$ of $\mathbb{R}^3$ by $\bar{\Psi}(x_1,x_2,x_3) = (\Psi(x_1,x_2),x_3)$. We define $Q'_R = \Psi(Q_R)$. Denoting by Ψ_i the i-th component of Ψ (where $i \in \{1,2\}$), we have, by the change of variable $(x'_1,x'_2) = \Psi(x_1,x_2)$,

$$\iint_{Q'_R} x'_i\, B_3[\vec{m},z](x'_1,x'_2)\,\mathrm{d}x'_1\mathrm{d}x'_2 = \iint_{Q_R} \Psi_i(\vec{x})\, B_3[\vec{m},z](\Psi(x_1,x_2))\,\mathrm{d}x_1\mathrm{d}x_2. \tag{26}$$

Moreover, according to Equation (1), we have, for any $\vec{x} \in \mathbb{R}^3$ such that $x_3 > r$,

$$B_3[\vec{m},x_3](\Psi(x_1,x_2)) = \frac{-\mu_0}{4\pi}\,\partial_{x_3}\left(\iiint_{\mathbb{R}^3} \frac{\langle \widetilde{\vec{m}}(\vec{t'}),\, \bar{\Psi}(\vec{x}) - \vec{t'}\rangle}{\|\bar{\Psi}(\vec{x}) - \vec{t'}\|^3}\,\mathrm{d}\vec{t'}\right).$$

Then, using the change of variable $\vec{t'} = \bar{\Psi}(\vec{t})$ and the fact that $\bar{\Psi}$ is a linear isometry (and hence preserves the inner product and the norm), the above expression becomes

$$\frac{-\mu_0}{4\pi}\,\partial_{x_3}\left(\iiint_{\mathbb{R}^3} \frac{\langle \bar{\Psi}^{-1}(\widetilde{\vec{m}}(\bar{\Psi}(\vec{t}))),\, \vec{x} - \vec{t}\rangle}{\|\vec{x} - \vec{t}\|^3}\,\mathrm{d}\vec{t}\right).$$

Finally, defining M_1, M_2 and M_3 by $\vec{M}(\vec{t}) = \bar{\Psi}^{-1}(\widetilde{\vec{m}}(\bar{\Psi}(\vec{t})))$, we observe that

$$B_3[\vec{m}, x_3](\Psi(x_1, x_2)) = B_3[\vec{M}, x_3](x_1, x_2). \tag{27}$$

Now, putting together Equations (26) and (27) and using the linearity of Ψ_1 and Ψ_2, we conclude that

$$\begin{pmatrix} \iint_{Q'_R} x'_1\, B_3[\vec{m}, z](x'_1, x'_2)\, \mathrm{d}x'_1 \mathrm{d}x'_2 \\ \iint_{Q'_R} x'_2\, B_3[\vec{m}, z](x'_1, x'_2)\, \mathrm{d}x'_1 \mathrm{d}x'_2 \end{pmatrix} = \Psi \begin{pmatrix} \iint_{Q_R} x_1\, B_3[\vec{M}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \\ \iint_{Q_R} x_2\, B_3[\vec{M}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \end{pmatrix}. \tag{28}$$

Accordingly,

$$\iint_{Q'_R} R\, B_3[\vec{m}, z](x'_1, x'_2)\, \mathrm{d}x'_1 \mathrm{d}x'_2 = \iint_{Q_R} R\, B_3[\vec{M}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{29}$$

We now express certain moments of M_1, M_2 and M_3. Let $i \in \{1, 2, 3\}$ and let us denote by $\bar{\Psi}_i^{-1}$ the i-th component of $\bar{\Psi}^{-1}$. Also, we consider an arbitrary bounded function $\theta : \mathbb{R} \to \mathbb{R}$. Using successively the definition of M_i, the change of variable $\vec{t'} = \bar{\Psi}(\vec{t})$ together with the fact that $\bar{\Psi}$ is a linear isometry, the fact that $\bar{\Psi}_3^{-1}(\vec{t'}) = t'_3$ and the linearity of $\bar{\Psi}_i^{-1}$, we get

$$\langle \theta(t_3)\, M_i \rangle = \iiint_{\mathbb{R}^3} \theta(t_3)\, \bar{\Psi}_i^{-1}(\widetilde{\vec{m}}(\bar{\Psi}(\vec{t})))\, \mathrm{d}\vec{t} = \iiint_{\mathbb{R}^3} \theta(\bar{\Psi}_3^{-1}(\vec{t'}))\, \bar{\Psi}_i^{-1}(\widetilde{\vec{m}}(\vec{t'}))\, \mathrm{d}\vec{t'} = \bar{\Psi}_i^{-1} \begin{pmatrix} \langle \theta(t_3)\, m_1 \rangle \\ \langle \theta(t_3)\, m_2 \rangle \\ \langle \theta(t_3)\, m_3 \rangle \end{pmatrix}.$$

Observing that, for any $\vec{x} \in \mathbb{R}^3$, $\bar{\Psi}^{-1}(\vec{x}) = (\Psi^{-1}(x_1, x_2), x_3)$, we thus obtain:

$$\begin{pmatrix} \langle \theta(t_3)\, M_1 \rangle \\ \langle \theta(t_3)\, M_2 \rangle \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \langle \theta(t_3)\, m_1 \rangle \\ \langle \theta(t_3)\, m_2 \rangle \end{pmatrix} \qquad \text{and} \qquad \langle \theta(t_3)\, M_3 \rangle = \langle \theta(t_3)\, m_3 \rangle. \tag{30}$$

Accordingly, we obtain $\langle t_i\, \theta(t_3)\, M_3 \rangle = \iiint_{\mathbb{R}^3} \bar{\Psi}_i^{-1}(\vec{t'})\, \theta(t'_3)\, \widetilde{m}_3(\vec{t'})\, \mathrm{d}\vec{t'} = \bar{\Psi}_i^{-1} \begin{pmatrix} \langle t_1\, \theta(t_3)\, m_3 \rangle \\ \langle t_2\, \theta(t_3)\, m_3 \rangle \\ \langle t_3\, \theta(t_3)\, m_3 \rangle \end{pmatrix}$, whence

$$\begin{pmatrix} \langle t_1\, \theta(t_3)\, M_3 \rangle \\ \langle t_2\, \theta(t_3)\, M_3 \rangle \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \langle t_1\, \theta(t_3)\, m_3 \rangle \\ \langle t_2\, \theta(t_3)\, m_3 \rangle \end{pmatrix} \qquad \text{and} \qquad \langle t_3\, \theta(t_3)\, M_3 \rangle = \langle t_3\, \theta(t_3)\, m_3 \rangle. \tag{31}$$

5.2 Proof of Equation (5)

To prove Equation (5), we define $\Psi : \vec{x} \mapsto (x_2, x_1)$. It is a linear involutive isometry, hence Equation (28) gives in this context

$$\iint_{Q'_R} x'_2\, B_3[\vec{m}, z](x'_1, x'_2)\, \mathrm{d}x'_1 \mathrm{d}x'_2 = \iint_{Q_R} x_1\, B_3[\vec{M}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2.$$

Now, observing that $Q'_R = Q_R$ and applying Equation (4) we get

$$\iint_{Q_R} x_2\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mu_0}{2} \langle M_1 \rangle + \frac{3\mu_0}{\pi R \sqrt{2}} \left(\langle t_1\, M_3 \rangle + \langle t_3\, M_1 \rangle - z \langle M_1 \rangle \right) + \mathcal{O}\Big(\frac{1}{R^3}\Big).$$

We conclude by remarking that $\langle M_1 \rangle = \langle m_2 \rangle$, $\langle t_1\, M_3 \rangle = \langle t_2\, m_3 \rangle$ and $\langle t_3\, M_1 \rangle = \langle t_3\, m_2 \rangle$ thanks to Equations (30), (31) with $\theta(t_3) = 1$ and Equation (30) with $\theta(t_3) = t_3 \cdot \chi_{[0,z]}(t_3)$.

5.3 Proofs of Equations (7), (8) and (9)

We define Ψ as the rotation of angle $\pi/4$ and we apply Equation (28) to $Q_{\frac{\sqrt{2}}{2}R}$, so that $Q'_{\frac{\sqrt{2}}{2}R} = S_R$. In this context, and according to Equations (4) and (5), Equation (28) becomes

$$\begin{pmatrix} \displaystyle\iint_{S_R} x_1\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \\ \displaystyle\iint_{S_R} x_2\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \end{pmatrix} = \Psi \begin{pmatrix} \dfrac{\mu_0}{2}\, \langle M_1 \rangle + \dfrac{3\mu_0}{\pi R}\, (\langle t_1\, M_3 \rangle + \langle t_3\, M_1 \rangle - z \langle M_1 \rangle) + \mathcal{O}\Big(\dfrac{1}{R^3}\Big) \\ \dfrac{\mu_0}{2}\, \langle M_2 \rangle + \dfrac{3\mu_0}{\pi R}\, (\langle t_2\, M_3 \rangle + \langle t_3\, M_2 \rangle - z \langle M_2 \rangle) + \mathcal{O}\Big(\dfrac{1}{R^3}\Big) \end{pmatrix}.$$

Using the linearity of Ψ, the right hand side of the above expression becomes

$$\frac{\mu_0}{2}\, \Psi \begin{pmatrix} \langle M_1 \rangle \\ \langle M_2 \rangle \end{pmatrix} + \frac{3\mu_0}{\pi R} \left(\Psi \begin{pmatrix} \langle t_1\, M_3 \rangle \\ \langle t_2\, M_3 \rangle \end{pmatrix} + \Psi \begin{pmatrix} \langle t_3\, M_1 \rangle \\ \langle t_3\, M_2 \rangle \end{pmatrix} - z\, \Psi \begin{pmatrix} \langle M_1 \rangle \\ \langle M_2 \rangle \end{pmatrix} \right) + \mathcal{O}\Big(\frac{1}{R^3}\Big),$$

which directly gives Equations (7) and (8) thanks to Equations (30) and (31) with $\theta(t_3) = 1$ and Equation (30) with $\theta(t_3) = t_3 \cdot \chi_{[0,s]}(t_3)$.

Now, together with Equation (6), Equation (29) gives us

$$\begin{aligned} \iint_{S_R} R\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 &= \frac{2\mu_0}{\pi} \langle M_3 \rangle \\ &+ \frac{5\mu_0}{2\pi R^2} (-2z^2 \langle M_3 \rangle + z\, (4\langle t_3\, M_3 \rangle - 2\langle t_1\, M_1 \rangle - 2\langle t_2\, M_2 \rangle) \\ &\qquad + \langle t_1^2 M_3 \rangle + \langle t_2^2 M_3 \rangle - 2\langle t_3^2 M_3 \rangle + 2\langle t_1 t_3\, M_1 \rangle + 2\langle t_2 t_3\, M_2 \rangle) + \mathcal{O}\Big(\frac{1}{R^3}\Big). \end{aligned}$$

Notice that (using again the change of variable $\vec{t} = \bar{\Psi}^{-1}(\vec{t}')$ and the fact that $\bar{\Psi}$ is a linear isometry)

$$\begin{aligned} \langle t_1^2\, M_3 \rangle + \langle t_2^2\, M_3 \rangle + \langle t_3^2\, M_3 \rangle &= \langle (t_1^2 + t_2^2 + t_3^2)\, M_3 \rangle \\ &= \iiint_{\mathbb{R}^3} \|\vec{t}\|^2\, \widetilde{m}_3(\bar{\Psi}(\vec{t}))\, \mathrm{d}\vec{t} \\ &= \iiint_{\mathbb{R}^3} \|\bar{\Psi}^{-1}(\vec{t}')\|^2\, \widetilde{m}_3(\vec{t}')\, \mathrm{d}\vec{t}' \\ &= \iiint_{\mathbb{R}^3} \|\vec{t}'\|^2\, \widetilde{m}_3(\vec{t}')\, \mathrm{d}\vec{t}' = \langle t_1^2\, m_3 \rangle + \langle t_2^2\, m_3 \rangle + \langle t_3^2\, m_3 \rangle. \end{aligned}$$

Combining this with Equation (30) where we set $\theta(t_3) = t_3^2 \cdot \chi_{[0,s]}(t_3)$, we get

$$\langle t_1^2\, M_3 \rangle + \langle t_2^2\, M_3 \rangle - 2\langle t_3^2\, M_3 \rangle = \langle t_1^2\, m_3 \rangle + \langle t_2^2\, m_3 \rangle - 2\langle t_3^2\, m_3 \rangle.$$

Also, if we denote by A the matrix of Ψ^{-1} in the canonical basis and by C_M (resp. C_m) the 3×3 matrix whose element (i, j) is $\langle t_i\, \theta(t_3)\, M_j \rangle$ (resp. $\langle t_i\, \theta(t_3)\, m_j \rangle$), we observe that

$$C_M = A\, C_m\, A^T. \tag{32}$$

Indeed, if i and j belong to $\{1, 2, 3\}$, then upon using the same change of variable we get that

$$\langle t_i\, \theta(t_3)\, M_j \rangle = \iiint_{\mathbb{R}^3} t_i\, \theta(t_3)\, \bar{\Psi}_j^{-1}(\widetilde{\vec{m}}(\bar{\Psi}(\vec{t})))\, \mathrm{d}\vec{t} = \iiint_{\mathbb{R}^3} \bar{\Psi}_i^{-1}(\vec{t}')\, \theta(t_3')\, \bar{\Psi}_j^{-1}(\widetilde{\vec{m}}(\vec{t}'))\, \mathrm{d}\vec{t}',$$

Denoting by A_i and A_j the i-th and j-th row of A respectively, we see that

$$\bar{\Psi}_i^{-1}(\vec{t}')\, \bar{\Psi}_j^{-1}(\widetilde{\vec{m}}(\vec{t}')) = (A_i\, \vec{t}')(A_j\, \widetilde{\vec{m}}(\vec{t}')) = (A_i\, \vec{t}')(A_j\, \widetilde{\vec{m}}(\vec{t}'))^T = A_i\, (\vec{t}'\, \widetilde{\vec{m}}(\vec{t}')^T)\, A_j^T.$$

Therefore, $\langle t_i\,\theta(t_3)\,M_j\rangle = A_i\,C_m\,A_j^T$ whence Equation (32) holds. Now, Ψ^{-1} being an isometry we have $A^T = A^{-1}$, and therefore

$$\begin{aligned}\langle t_1\,\theta(t_3)\,M_1\rangle + \langle t_2\,\theta(t_3)\,M_2\rangle + \langle t_3\,\theta(t_3)\,M_3\rangle &= \operatorname{tr}(A\,C_m\,A^T)\\ &= \operatorname{tr}(A^T A\,C_m)\\ &= \operatorname{tr}(C_m)\\ &= \langle t_1\,\theta(t_3)\,m_1\rangle + \langle t_2\,\theta(t_3)\,m_2\rangle + \langle t_3\,\theta(t_3)\,m_3\rangle.\end{aligned}$$

Together with Equation (31), this shows that $\langle t_1\,\theta(t_3)\,M_1\rangle + \langle t_2\,\theta(t_3)\,M_2\rangle = \langle t_1\,\theta(t_3)\,m_1\rangle + \langle t_2\,\theta(t_3)\,m_2\rangle$. We conclude the proof, using that last result with $\theta(t_3) = 1$ and $\theta(t_3) = t_3\,\chi_{[0,s]}(t_3)$.

6 Comments, discussion

Notice that these bounds also depend on the quantities $\langle |m_i|\rangle$ $(i = 1, 2, 3)$.

Remark 2. For the purpose of proving our result in the most general framework, we introduced the constants ω_s and ω_z, so as to determine the constants usually hidden behind the $\mathcal{O}$ notation and make these constants explicit as functions depending only on s and z but not on $\vec{t}$ and R. This allows us to integrate these bounds with respect to variable $\vec{t}$ and obtain upper bounds for δ_{18} and δ_{23} (at the end of Sections 4.2, 4.3, respectively), therefore proving rigorous asymptotic formulas that give approximate identities more and more accurate as R goes large.

However, the practice is usually completely different from this situation. One generally does not actually let R tends to $+\infty$, but one rather has some measurements on a square Q_R with a given value R and one would like to get an approximation of the moments $\langle m_i\rangle$ $(i = 1, 2, 3)$ using the asymptotic formulas, together with an estimate of the error contained in the remainder. In order to obtain a small bound for the remainder, it is clearly desirable to choose ω_s and ω_z as small as possible, so one practically chooses $\omega_s = s/R$ and $\omega_z = z/R$ (as soon as $R > s$). Therefore Equation (15) defines the constant C as being equal to R, see also Remark 1.

Furthermore, instead of using the interval $[-1 + (1-\omega_s)^2, 2\omega_s + \omega_s^2 + \omega_z^2]$ in Lemma 4, one can use slightly tighter intervals since all the constants are known. For instance, Equations (38) and (39) can be reworked to show that $u_1(t_1, t_2, (z - t_3), 1/R)$ indeed ranges in the interval $[-1 + \frac{(z-r)^2}{2R^2} + (1 - \frac{s}{R})^2,\ \frac{2s}{R} + \frac{2s^2+z^2}{2R^2}]$ for any $\vec{t} \in \mathcal{A}$. Also, notice that explicit tight and rigorous constants $B^{(1)}$ to $B^{(5)}$ for Lemma 4 can be automatically computed on demand for a given interval using rigorous arithmetic tools such as Taylor models [8]. Together with the present article, we provide a Maple script that explicitly computes all the presented bounds, for given values s, z and R. For the computation of the bounds of Lemma 4, we rest on a script run with the Sollya software tool [3] that provides rigorous and proven results, accounting for all possible roundoff errors in numeric computations.

The main results (4) to (9) of Section 3 can be restated on the measurement area $M_R \in \{Q_R, S_R\}$ as:

$$\iint_{M_R} x_i\,B_3[\vec{m}, z](x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = c_{0,i}^{(1)}(M_R)\,\langle m_i\rangle + \frac{c_{1,i}^{(1)}(M_R)}{R}\left(\langle t_i\,m_3\rangle + \langle (t_3 - z)\,m_i\rangle\right) + \mathcal{O}\Big(\frac{1}{R^3}\Big),$$

for $i = 1, 2$ and:

$$\begin{aligned}\iint_{M_R} B_3[\vec{m}, z](x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = &\frac{c_{1,3}^{(0)}(M_R)}{R}\langle m_3\rangle\\ &+\frac{c_{3,3}^{(0)}(M_R)}{R^3}\Big(\langle (t_1^2 + t_2^2 - 2\,(t_3 - z)^2)\,m_3\rangle - 2\,\langle (t_3 - z)\,(t_1\,m_1 + t_2\,m_2)\rangle\Big) + \mathcal{O}\Big(\frac{1}{R^4}\Big),\end{aligned}$$

with

$$c_{0,i}^{(1)}(S_R) = \frac{\mu_0}{2}\,,\; c_{1,i}^{(1)}(S_R) = \frac{3\mu_0}{\pi}\,,\; c_{1,3}^{(0)}(S_R) = \frac{2\mu_0}{\pi}\,,\; c_{3,3}^{(0)}(S_R) = \frac{5\mu_0}{2\pi}\,,$$

and, for $j = 0, 1$,

$$c_{k,i}^{(j)}(Q_R) = \frac{c_{k,i}^{(j)}(S_R)}{2^{k/2}} = \frac{c_{k,i}^{(j)}(S_R)}{\sqrt{2}^k} \text{ (while } c_{2,i}^{(1)}(M_R) = c_{0,3}^{(0)}(M_R) = c_{2,3}^{(0)}(M_R) = 0)\,.$$

Actually, $\sqrt{2}$ appears here because it is the ratio between the perimeters of the measurements areas Q_R and S_R. Similar properties appear to be true for circular shapes as well [9, Part III], and most probably also for more general geometries. Recall that $B_3[\vec{m}, z]$ has vanishing mean value on $\mathbb{R}^2$ by Green's theorem, which is consistent with the last equality above and with Equations (6) and (9).

This allows one to algebraically combine between the two families of expressions obtained for two different shapes in order to refine the estimates of $\langle m_i \rangle$ that could be obtained using a single shape. For instance, we get for $i = 1, 2$:

$$\frac{(\sqrt{2}-1)\,\mu_0}{2}\,\langle m_i \rangle = \sqrt{2}\iint_{Q_R} x_i\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1\mathrm{d}x_2 - \iint_{S_R} x_i\, B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1\mathrm{d}x_2 + \mathcal{O}\Big(\frac{1}{R^3}\Big)\,,$$

$$\frac{2\,\mu_0}{\pi\, R}\,\langle m_3 \rangle = 2\sqrt{2}\iint_{Q_R} B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1\mathrm{d}x_2 - \iint_{S_R} B_3[\vec{m}, z](x_1, x_2)\, \mathrm{d}x_1\mathrm{d}x_2 + \mathcal{O}\Big(\frac{1}{R^4}\Big)\,.$$

Whenever R is such that Q_R and S_R are subsets of the actual measurement area, these furnish constructive approximations of $\langle m_i \rangle$. The precision of these estimations is driven by the size R of the measurement area, but also by the heights s, r, z. Indeed, the error terms decrease to 0 not only when R increases but also when s, r, z (and $r - z$) decrease, as the analysis of the bounds in the preceding section and in the appendix shows.

7 Conclusion

We derived formulas for estimating the net moment of compactly supported magnetization distributions from measurements of the component of its associated magnetic field that is normal to the measurement plane. Those expressions can be used to essentially transform any magnetometer or technique that images the magnetic field of a magnetization distribution on a plane into a moment magnetometer. Although our main motivation was to develop a net moment estimation technique for scanning magnetic microscopy data, our formulas can be easily adapted to operate on other vector magnetic field component or (scalar) total field measurements. A key feature in our technique is that we make minimal assumptions about source geometry and characteristics, which makes it useful to a broad range of applications.

Whereas several issues of practical interest may negatively impact the quality of the moment estimates and remain to be studied in more detail, we have nevertheless provided a set of formulas that constitute the basis of net moment estimation for planar measurement geometries. Some of those issues are: (a) data discretization – magnetic data are only available at discrete points on a rectangular grid, which leads one to approximate the integrals in Equations (4), (5), (6), (7), (8), and (9) by Riemann sums, thereby making for another approximation step; (b) noisy data – the measurements themselves are often corrupted by noise whose effect on the estimates has to be systematically analyzed; (c) size of measurement area compared to magnetization support…

For instance, it should be observed that our estimates of hm1i, hm2i are unbiased, while it is the linear combination of equations proposed in Section 6 that can gets us rid of the bias in our estimates of hm3i.

• Following Section 3: however, restricted areas of measurement, $R < 1$ may be not that large $\rightsquigarrow$ BEP, work in preparation.

Appendix: proof of Lemma 1

We first establish three lemmas about Taylor expansions of order 2 that will be used for the proof of Lemma 1.

As before in Section 4, we consider two positive constants ω_s and ω_z such that $\omega_s < 1$ and we define C as in Equation (15). In the sequel, we will extensively use the notations defined in Equations (16) and (18), together with their practical properties, as given by Equations (17) and (19).

Put $\mathcal{D} = \mathcal{D}(s, z, C) = [-s, s]^2 \times (0, z] \times [0, 1/C]$. The assumption $(\vec{\gamma}, \xi) \in \mathcal{D}$ actually means $\vec{\gamma} \in [-s, s]^2 \times (0, z]$, $\xi \in [0, 1/C]$.

The following lemma explains how to compose Taylor expansions of order 2 with rigorous bounds.

Lemma 2. *We consider an interval I, a function $v : I \to \mathbb{R}$ and a function $u : \mathcal{D} \to I$ and we suppose that they are of the form*

$$\begin{cases} v(y) = b_0 + b_1 y + b_2 y^2 + \varepsilon_v(y) \qquad \text{with} \qquad \forall y \in I,\ |\varepsilon_v(y)| \le B_3\,|y|^3, \\ u(\vec{\gamma}, \xi) = a_1(\vec{\gamma})\xi + a_2(\vec{\gamma})\xi^2 + \varepsilon_u(\vec{\gamma}, \xi) \\ \qquad\qquad\qquad\qquad \text{with } \forall(\vec{\gamma}, \xi) \in \mathcal{D},\ |\varepsilon_u(\vec{\gamma}, \xi)| \le A_3(s, z)\xi^3, \end{cases}$$

where $b_0, b_1, b_2, B_3 \in \mathbb{R}$, and $a_1(\vec{\gamma})$, $a_2(\vec{\gamma})$, $A_3(s, z)$ are homogeneous polynomials of degree 1, 2, 3, respectively.

Then, we have $(v \circ u)(\vec{\gamma}, \xi) = c_0 + c_1(\vec{\gamma})\xi + c_2(\vec{\gamma})\xi^2 + \varepsilon(\vec{\gamma}, \xi)$ with $\forall(\vec{\gamma}, \xi) \in \mathcal{D}$, $|\varepsilon(\vec{\gamma}, \xi)| \le C_3(s, z)\xi^3$, where $c_0 = b_0$, $c_1(\vec{\gamma}) = b_1 a_1(\vec{\gamma})$, $c_2(\vec{\gamma}) = b_1 a_2(\vec{\gamma}) + b_2 a_1^2(\vec{\gamma})$, and $C_3(s, z)$ is a homogeneous polynomial of degree 3.

Proof. Let $(\vec{\gamma}, \xi) \in \mathcal{D}$. From the definitions of u and v, we get

$$\varepsilon(\vec{\gamma}, \xi) = b_1\ \varepsilon_u(\vec{\gamma}, \xi) + b_2\Big(2a_1(\vec{\gamma})\xi + a_2(\vec{\gamma})\xi^2 + \varepsilon_u(\vec{\gamma}, \xi)\Big)\Big(a_2(\vec{\gamma})\xi^2 + \varepsilon_u(\vec{\gamma}, \xi)\Big) + \varepsilon_v(u(\vec{\gamma}, \xi)).$$

The bound then follows from the triangle inequality, with

$$\begin{aligned} C_3(s, z) = |b_1|A_3(s, z) + |b_2|\Big(2A_1(s, z) + A_{2,1}(s, z) + A_{3,2}(s, z)\Big)\Big(A_2(s, z) + A_{3,1}(s, z)\Big) \\ + B_3\left(A_1(s, z) + A_{2,1}(s, z) + A_{3,2}(s, z)\right)^3, \end{aligned}$$

and the definitions of A_1, A_2, $A_{2,1}$, $A_{3,1}$, and $A_{3,2}$ from $a_1(\vec{\gamma})$, $a_2(\vec{\gamma})$, and $A_3(s, z)$, by means of Equations (16) and (18), together with the corresponding properties expressed in Equations (17) and (19). □

In particular, notice that $c_1(\vec{\gamma})$, $c_2(\vec{\gamma})$ are homogeneous polynomials of degree 1 and 2, respectively, as in the following lemma which, in the same spirit, shows how to multiply Taylor expansions of order 2 with rigorous bounds.

Lemma 3. *We consider two functions u and $v : \mathcal{D} \to \mathbb{R}$ and we suppose that they are of the form*

$$\begin{cases} u(\vec{\gamma}, \xi) = 1 + a_1(\vec{\gamma})\xi + a_2(\vec{\gamma})\xi^2 + \varepsilon_u(\vec{\gamma}, \xi) \\ v(\vec{\gamma}, \xi) = 1 + b_1(\vec{\gamma})\xi + b_2(\vec{\gamma})\xi^2 + \varepsilon_v(\vec{\gamma}, \xi) \end{cases}$$

with $\forall(\vec{\gamma}, \xi) \in \mathcal{D}$, $|\varepsilon_u(\vec{\gamma}, \xi)| \leq A_3(s, z)\xi^3$ and $|\varepsilon_v(\vec{\gamma}, \xi)| \leq B_3(s, z)\xi^3$, where $a_1(\vec{\gamma})$, $b_1(\vec{\gamma})$, $a_2(\vec{\gamma})$, $b_2(\vec{\gamma})$, and $A_3(s, z)$, $B_3(s, z)$ are homogeneous polynomials of degree 1, 2, and 3, respectively.

Then, we have

$$u(\vec{\gamma}, \xi)v(\vec{\gamma}, \xi) = 1 + c_1(\vec{\gamma})\xi + c_2(\vec{\gamma})\xi^2 + \varepsilon(\vec{\gamma}, \xi)$$

with $\forall(\vec{\gamma}, \xi) \in \mathcal{D}$, $|\varepsilon(\vec{\gamma}, \xi)| \leq C_3(s, z)\xi^3$, where $c_1(\vec{\gamma}) = a_1(\vec{\gamma}) + b_1(\vec{\gamma})$, $c_2(\vec{\gamma}) = a_1(\vec{\gamma})b_1(\vec{\gamma}) + a_2(\vec{\gamma}) + b_2(\vec{\gamma})$, and $C_3(s, z)$ is a homogeneous polynomial of degree 3.

Proof. Let $(\vec{\gamma}, \xi) \in \mathcal{D}$. From the definitions of u and v, we get

$$\begin{aligned} \varepsilon(\vec{\gamma}, \xi) = \varepsilon_u(\vec{\gamma}, \xi) + \varepsilon_v(\vec{\gamma}, \xi) \\ + \Big(a_1(\vec{\gamma})\xi + \tfrac{1}{2}\, a_2(\vec{\gamma})\xi^2 + \tfrac{1}{2}\, \varepsilon_u(\vec{\gamma}, \xi)\Big)\Big(b_2(\vec{\gamma})\xi^2 + \varepsilon_v(\vec{\gamma}, \xi)\Big) \\ + \Big(b_1(\vec{\gamma})\xi + \tfrac{1}{2}\, b_2(\vec{\gamma})\xi^2 + \tfrac{1}{2}\, \varepsilon_v(\vec{\gamma}, \xi)\Big)\Big(a_2(\vec{\gamma})\xi^2 + \varepsilon_u(\vec{\gamma}, \xi)\Big). \end{aligned}$$

The bound then again follows from the triangle inequality, with

$$\begin{aligned} C_3(s, z) = A_3(s, z) + B_3(s, z) \\ + \Big(A_1(s, z) + \tfrac{1}{2}\, A_{2,1}(s, z) + \tfrac{1}{2}\, A_{3,2}(s, z)\Big)\Big(B_2(s, z) + B_{3,1}(s, z)\Big) \\ + \Big(B_1(s, z) + \tfrac{1}{2}\, B_{2,1}(s, z) + \tfrac{1}{2}\, B_{3,2}(s, z)\Big)\Big(A_2(s, z) + A_{3,1}(s, z)\Big), \end{aligned}$$

and the definitions of A_1, A_2, $A_{2,1}$, $A_{3,1}$, $A_{3,2}$, B_1, B_2, $B_{2,1}$, $B_{3,1}$, and $B_{3,2}$ from $a_1(\vec{\gamma})$, $a_2(\vec{\gamma})$, $A_3(s, z)$, $b_1(\vec{\gamma})$, $b_2(\vec{\gamma})$, and $B_3(s, z)$, by means of Equations (16) and (18), together with the corresponding properties expressed in Equations (17) and (19). □

The following lemma collects some Taylor expansions with controlled bounds on their remainders. They will be used in the proof of Lemma 1.

Lemma 4. *We define the functions v_1 to v_5 by $v_1(y) = \frac{1}{\sqrt{1+y}}$, $v_2(y) = \frac{1}{1+y}$, $v_3(y) = \sqrt{1+y}$, $v_4(y) = \arctan(y)$ and $v_5(y) = \operatorname{arcsinh}(1 + y)$. There exist constants $B^{(1)}, \cdots, B^{(5)}$, depending on ω_s and ω_z only, such that*

$\forall y \in [-1 + (1 - \omega_s)^2,\, 2\omega_s + \omega_s^2 + \omega_z^2]$,

$$v_1(y) = 1 - \frac{y}{2} + \frac{3y^2}{8} + \varepsilon_1(y) \quad \textit{with} \quad |\varepsilon_1(y)| \leq B^{(1)}|y|^3, \tag{33}$$

$$v_2(y) = 1 - y + y^2 + \varepsilon_2(y) \quad \textit{with} \quad |\varepsilon_2(y)| \leq B^{(2)}|y|^3, \tag{34}$$

$$v_3(y) = 1 + \frac{y}{2} - \frac{y^2}{8} + \varepsilon_3(y) \quad \textit{with} \quad |\varepsilon_3(y)| \leq B^{(3)}|y|^3, \tag{35}$$

$\forall y \in \mathbb{R}$,

$$v_4(y) = y + \varepsilon_4(y) \quad \textit{with} \quad |\varepsilon_4(y)| \leq B^{(4)}|y|^3, \tag{36}$$

$$v_5(y) = \operatorname{arcsinh}(1) + \frac{y}{\sqrt{2}} - \frac{y^2}{4\sqrt{2}} + \varepsilon_5(y) \quad \textit{with} \quad |\varepsilon_5(y)| \leq B^{(5)}|y|^3. \tag{37}$$

Proof. If $i \in \{1, 2, 3\}$, since $\omega_s < 1$, the function v_i is infinitely differentiable on the given interval and the existence of $B^{(i)}$ simply follows from Taylor's theorem at 0. If $i \in \{4, 5\}$, the same argument applied on the interval $[-1, 1]$ ensures the existence of a constant $B'^{(i)}$ such that Equations (36) and (37) hold true for $y \in [-1, 1]$. Besides, we observe that the function $y \mapsto \varepsilon_i(y)/y^3$ is continuous on $(-\infty, -1] \cup [1, +\infty)$ and tends to 0 at $\pm\infty$. Therefore, there exists a constant $B''^{(i)}$ such that Equations (36) and (37) hold true for $y \in (-\infty, -1] \cup [1, +\infty)$. Then $B^{(i)} = \max\{B'^{(i)}, B''^{(i)}\}$ satisfies the requirement. $\square$

Proof of Lemma 1. Let us define

$$u_1(\vec{\gamma}, \xi) = -(\gamma_1 + \gamma_2)\xi + \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2}\,\xi^2 = -1 + \frac{\gamma_3^2\xi^2}{2} + \frac{(1-\gamma_1\xi)^2}{2} + \frac{(1-\gamma_2\xi)^2}{2}.$$

According to Equation (15), observe that for $(\vec{\gamma}, \xi) \in \mathcal{D}$, $\gamma_1\xi \leq |\gamma_1\xi| \leq s\xi \leq s/C \leq \omega_s < 1$. Therefore $(1-\gamma_1\xi) \geq (1-\omega_s) > 0$, hence $(1-\gamma_1\xi)^2 \geq (1-\omega_s)^2$. Similarly, $(1-\gamma_2\xi)^2 \geq (1-\omega_s)^2$. This shows that

$$u_1(\vec{\gamma}, \xi) \geq -1 + \frac{\gamma_3^2\xi^2}{2} + (1-\omega_s)^2 \geq -1 + (1-\omega_s)^2. \tag{38}$$

Besides, from the definition of u_1, the triangle inequality and Equation (15) we obtain

$$u_1(\vec{\gamma}, \xi) \leq |u_1(\vec{\gamma}, \xi)| \leq \frac{2s}{C} + \frac{2s^2 + z^2}{2C^2} \leq 2\omega_s + \omega_s^2 + \frac{\omega_z^2}{2}. \tag{39}$$

We also define

$$u_2(\vec{\gamma}, \xi) = -2\gamma_1\xi + (\gamma_1^2 + \gamma_3^2)\xi^2 = -1 + \gamma_3^2\xi^2 + (1-\gamma_1\xi)^2.$$

Reasoning as we did for u_1, we see that for $(\vec{\gamma}, \xi) \in \mathcal{D}$,

$$\begin{aligned} u_2(\vec{\gamma}, \xi) &\geq -1 + (1-\omega_s)^2, && (40)\\ u_2(\vec{\gamma}, \xi) &\leq |u_2(\vec{\gamma}, \xi)| \leq 2\omega_s + \omega_s^2 + \omega_z^2. && (41) \end{aligned}$$

Finally, we define $u_3(\vec{\gamma}, \xi) = -(\gamma_1+\gamma_2)\xi + \gamma_1\gamma_2\xi^2 = -1 + (1-\gamma_1\xi)(1-\gamma_2\xi)$, for which, accordingly,

$$\begin{aligned} u_3(\vec{\gamma}, \xi) &\geq -1 + (1-\omega_s)^2, && (42)\\ u_3(\vec{\gamma}, \xi) &\leq |u_3(\vec{\gamma}, \xi)| \leq 2\omega_s + \omega_s^2. && (43) \end{aligned}$$

Now, for any $\vec{t} \in \mathcal{A}$ and any $R \geq C$, we have

$$\begin{aligned} (R-t_1)^2 + (R-t_2)^2 + (z-t_3)^2 &= 2R^2\,(1 + u_1(\vec{\gamma}, \xi)),\\ (R-t_1)^2 + (z-t_3)^2 &= R^2\,(1 + u_2(\vec{\gamma}, \xi)),\\ (R-t_1)(R-t_2) &= R^2\,(1 + u_3(\vec{\gamma}, \xi)), \end{aligned}$$

where $\vec{\gamma} = (t_1, t_2, (z-t_3))$ and $\xi = 1/R$. Therefore, the functions $F_{\vec{t}}$, $G_{\vec{t}}$, $K_{\vec{t}}$, and $L_{\vec{t}}$ can be expressed by means of v_1 to v_5 and u_1 to u_3. Namely (for $K_{\vec{t}}$ we use the identity $\arctan(\alpha) = \frac{\pi}{2} - \arctan(1/\alpha)$ which holds for $\alpha > 0$):

$$\begin{aligned} F_{\vec{t}}(R) &= \frac{\xi^2}{\sqrt{2}}\,(1-\gamma_2\xi)\,v_2(u_2(\vec{\gamma}, \xi))\,v_1(u_1(\vec{\gamma}, \xi)),\\ G_{\vec{t}}(R) &= \frac{-\xi}{\sqrt{2}}\,v_1(u_1(\vec{\gamma}, \xi)),\\ K_{\vec{t}}(R) &= \frac{\pi}{2\gamma_3} - \frac{1}{\gamma_3}v_4\left(\gamma_3\xi\sqrt{2}\,v_3(u_1(\vec{\gamma}, \xi))\,v_2(u_3(\vec{\gamma}, \xi))\right),\\ L_{\vec{t}}(R) &= -v_5\left(-1 + (1-\gamma_2\xi)\;v_1(u_2(\vec{\gamma}, \xi))\right). \end{aligned}$$

Let us recall the statement that we wish to prove, namely that for any $R \geq C$ and $\vec{t} \in \mathcal{A}$ we ave:

$$F_{\vec{t}}(R) = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10t_3^2 + 20zt_3 - 10z^2}{8\sqrt{2}} \cdot \frac{1}{R^4} + \frac{\delta_1(\vec{t}, z, R)}{R^5},$$

$$G_{\vec{t}}(R) = \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} - \frac{t_1^2 + t_2^2 + 6t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{8\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_2(\vec{t}, z, R)}{R^4},$$

$$K_{\vec{t}}(R) = \frac{\pi}{2(z - t_3)} - \sqrt{2}\,\frac{1}{R} - \frac{\sqrt{2}(t_1 + t_2)}{2} \cdot \frac{1}{R^2} + \frac{\delta_3(\vec{t}, z, R)}{R^3},$$

$$L_{\vec{t}}(R) = -\operatorname{arcsinh}(1) + \frac{t_2 - t_1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{3t_1^2 - t_2^2 - 2t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{4\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_4(\vec{t}, z, R)}{R^3},$$

where $|\delta_1(\vec{t}, z, R)| \leq \Delta_3^{(1)}(s, z)$, $|\delta_2(\vec{t}, z, R)| \leq \Delta_3^{(2)}(s, z)$, $|\delta_3(\vec{t}, z, R)| \leq \Delta_2^{(3)}(s, z)$, and $|\delta_4(\vec{t}, z, R)| \leq \Delta_3^{(4)}(s, z)$, for some homogeneous polynomials $\Delta_3^{(1)}$, $\Delta_3^{(2)}$, $\Delta_2^{(3)}$, and $\Delta_3^{(4)}$, of degrees 3, 3, 2, and 3, respectively. Except for $K_{\vec{t}}$ (which requires some explanation), the statement is easily deduced from our previous results, using Lemma 2 with $I = [-1 + (1 - \omega_s)^2, 2\omega_s + \omega_s^2 + \omega_z^2]$ for the compositions involving v_1, v_2, and v_3, and $I = (-\infty, +\infty)$ for the compositions involving v_4 and v_5, together with Lemmas 3 and 4.

In the case of $K_{\vec{t}}$ a difficulty arises from the division by $\gamma_3 = z - t_3$ which casts doubt on whether $\Delta_2^{(3)}(s, z)$ can be chosen as a polynomial. Let us put $d(\vec{\gamma}, \xi) = \sqrt{2}\, v_3(u_1(\vec{\gamma}, \xi))\, v_2(u_3(\vec{\gamma}, \xi))$ whence $K_{\vec{t}}(R) = \frac{\pi}{2\gamma_3} - \frac{1}{\gamma_3}\, v_4(\gamma_3\, \xi\, d(\vec{\gamma}, \xi))$. Thanks to our lemmas, we easily obtain that

$$d(\vec{\gamma}, \xi) = \sqrt{2} + d_1(\vec{\gamma})\xi + d_2(\vec{\gamma})\xi^2 + \varepsilon_d(\vec{\gamma}, \xi),$$

where d_1 and d_2 are homogeneous polynomials of degree 1 and 2 respectively, and $|\varepsilon_d(\vec{\gamma}, \xi)| \leq D_3(s, z)\xi^3$ where D_3 is a homogeneous polynomial of degree 3. Now, from Equation (36) we see that $K_{\vec{t}}$ has the prescribed form with the remainder $\delta_3(\vec{t}, z, R)/R^3$ being given by

$$d_2(\vec{\gamma})\xi^3 + \varepsilon_d(\vec{\gamma}, \xi)\xi + \frac{1}{\gamma_3}\, \varepsilon_4(\gamma_3\, \xi\, d(\vec{\gamma}, \xi)),$$

with, as before, $\vec{\gamma} = (t_1, t_2, (z - t_3))$ and $\xi = 1/R$. Now, observe that $|d_2(\vec{\gamma})\xi^3| \leq D_2(s, z)\xi^3$, $|\varepsilon_d(\vec{\gamma}, \xi)\xi| \leq D_{3,1}(s, z)\xi^3$ and

$$\left|\frac{1}{\gamma_3}\, \varepsilon_4(\gamma_3\, \xi\, d(\vec{\gamma}, \xi))\right| \leq \frac{1}{|\gamma_3|}\, B^{(4)} |\gamma_3|^3\, |\xi|^3\, d(\vec{\gamma}, \xi)^3 \leq B^{(4)}\, z^2\, (\sqrt{2} + D_{1,1} + D_{2,2} + D_{3,3})^3$$

which proves the claim. □

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